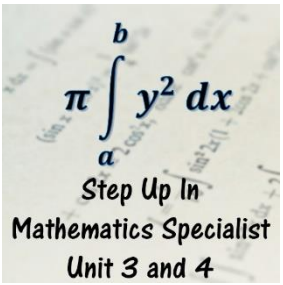
3.6 Vector Calculus

Problems Worksheet



1. Differentiate the following expressions with respect to time.

   a. $\boldsymbol{r}(\boldsymbol{t}) = e^t\boldsymbol{i} + 2t\cos t\,\boldsymbol{j}\ m$

   b. $\boldsymbol{v}(\boldsymbol{t}) = \sin^2 t\,\boldsymbol{i} - \frac{1}{\sqrt{t+1}}\boldsymbol{j}\ m/s$

   c. $\boldsymbol{r}(\boldsymbol{t}) = \tan 2t\,\boldsymbol{i} + (3t-\pi)\boldsymbol{j} - (t^3 - \sin t)\boldsymbol{k}\ m$

2. Use the information provided to determine an expression for the position of the objects below as a function of time.

   a. $\boldsymbol{v}(t) = 6\cos 2t\,\boldsymbol{i} - 2\sin t\,\boldsymbol{j}\ m/s$, and $\boldsymbol{r_0} = 2\boldsymbol{i} + 5\boldsymbol{j}\ m$.

   b. $\boldsymbol{a}(t) = -9.8\boldsymbol{j}\ m/s^2$, and $\boldsymbol{r_0} = 0\boldsymbol{i} + 0\boldsymbol{j}\ m$, and $\boldsymbol{v_0} = 3\boldsymbol{i} + 14\boldsymbol{j}\ m/s$.

   c. $\boldsymbol{v}(t) = -\cos t\,\boldsymbol{i} + (4 - \sin^2 2t)\boldsymbol{j}$, and $\boldsymbol{r_0} = 3\boldsymbol{i} + 3\boldsymbol{j}\ m$

   d. $\boldsymbol{v}(t) = \sin t\,\boldsymbol{i} + (2 - \sin t)\boldsymbol{j} + (\cos^2 t - 3\sin^2 t + 1)\boldsymbol{k}\ m/s$, and $\boldsymbol{r_0} = 4\boldsymbol{i} - 2\boldsymbol{j} + 10\boldsymbol{k}\ m$

3. Consider a projectile moving with the following position vectors as a function of time. In each case, the time is in seconds and the position in metres. Determine:
   i. the total distance travelled during the time interval specified;
   ii. the displacement between the time intervals specified; and
   iii. the magnitude of the displacement between the time intervals specified.

   a. $\boldsymbol{r}(t) = \cos t\, \boldsymbol{i} + \sin t\, \boldsymbol{j}$, $0 \leq t \leq \pi$.

   b. $\boldsymbol{r}(t) = \cos t\, \boldsymbol{i} + \sin t\, \boldsymbol{j}$, $0 \leq t \leq 2\pi$.

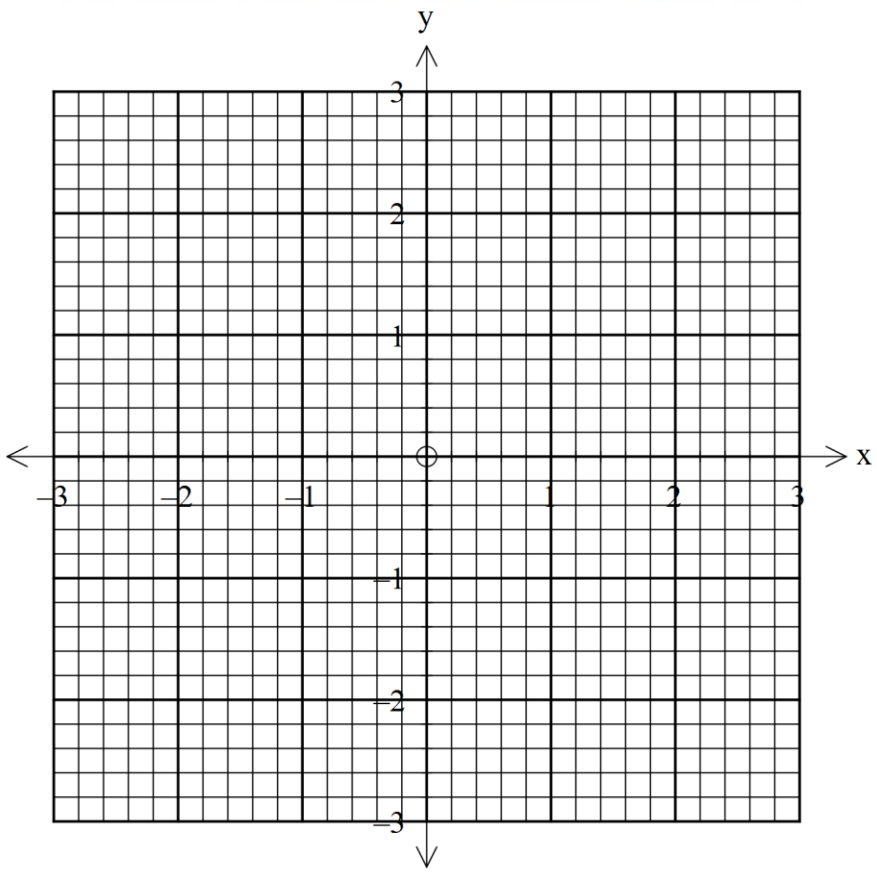
   c. $\boldsymbol{r}(t) = 0.1e^{3t}\boldsymbol{i} - 2(t - \sin t)\boldsymbol{j}$, $0 \leq t \leq 2$.

   d. $\boldsymbol{r}(t) = 2t \sin 2t\, \boldsymbol{i} + \frac{t-1}{t+1}\boldsymbol{j}$, $1 \leq t \leq 2$

4. Consider a body moving with displacement $\boldsymbol{r}(t) = 2\sin t\,\boldsymbol{i} + 3\cos 3t\,\boldsymbol{j}\ m$, with time t in seconds. The direction of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are in the positive x and y directions respectively.

   a. Calculate the period of the body's movement.

   b. Sketch the path of the object for one period of its motion.



   c. Upon your sketch in part (b), indicate the velocity of the body at time $t = \frac{3\pi}{4}$.

   d. Calculate the distance travelled by the body for $0 \leq t \leq \frac{3\pi}{4}$. Give your answer to two decimal places.

   e. Calculate the magnitude of the displacement of the body for $0 \leq t \leq \frac{3\pi}{4}$ with the use of an integral. Give your answers to two decimal places

5. A projectile is launched from the edge of a 100 m high cliff with velocity $\boldsymbol{v_0} = 12\boldsymbol{i} + 7\boldsymbol{j}\ m/s$ where the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ represent the forward and upward directions respectively. Acceleration due to gravity is 9.80 m s^{-2}. There is no motion in the lateral direction of the projectile and there is no air resistance. Using the techniques of vector calculus, determine:

    a. An expression for the position of the projectile for time t.

    b. The time at which the particle's height is a maximum, and that maximum height above the base of the cliff.

    c. The total distance (nearest centimetre) travelled by the projectile at the instant it strikes the ground.

    d. The magnitude of the displacement (nearest centimetre) of the projectile at the instant it strikes the ground. In determining this value, make use of a suitable integral.

    e. The velocity of the projectile as it strikes the ground.

    f. The speed of the projectile as it strikes the ground.

    g. The time(s) during the flight of the projectile when its speed is zero.

6. A figure skater is moving across the ice. Her routine is such that as she moves, her position is given by the vector equation $\boldsymbol{r}(t) = -15\sin\frac{t}{3}\boldsymbol{i} - 20\sin\frac{t}{4}\boldsymbol{j}\ m$, where the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are measured in the north and east directions from the centre of the rectangular rink, and time t is in seconds.

    a. Calculate the period, T, of the skater's motion.

    b. Show that the speed of the skater at $t = \pi\ s$ is $\frac{5\sqrt{3}}{2}\ m/s$.

    c. Calculate the length one complete lap of the skater's circuit, to the nearest metre.

7. Note: This question goes beyond the syllabus as it is a three-dimensional problem. However, if you can solve this problem then you have an excellent grasp of this topic.

   Jack Billings of the St Kilda Saints is taking a set shot after the final siren, to win the 2018 AFL Grand Final against the West Coast Eagles. He is standing directly in front of the centre of the goal posts and will kick from 65.6 m out, whilst holding the ball 0.5 m above the ground. Due to a light breeze at the ground, and to ensure he makes the distance, he imparts a velocity of $-1.1\boldsymbol{i} + 16\boldsymbol{j} + +20.8\boldsymbol{k}\ m/s$ and with a spin accelerating the ball with magnitude $0.3\boldsymbol{i}\ m/s^2$. Take the acceleration due to gravity to be 9.8 m/s^2 towards the Earth. The $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ vectors represent the right, forward and upward directions from Billings' perspective.

   a. Use the information provided to state the initial velocity and position vectors of the ball, and the acceleration of the ball.

   b. Determine an expression for the position of the ball as a function of time.

   c. Calculate the amount of time that passes between the ball being kicked and it crossing the goal line (the line between the two goal posts).

   d. Determine whether the kick will be a goal. I.e., that it passes between the goal posts, given the goal posts are 6.4 m apart.

   e. Members of the opposition team are standing on the goal line. As the ball passes, they will jump and attempt to touch the ball, meaning a goal will not be scored. Assuming these players can jump and reach a maximum height of 3.2 m, will the kick still be a goal?

   f. What is the distance travelled by the ball between the time it is kicked and the time it reaches the goal line?